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Can Pions “Smell” 4D, $N = 1$ Supersymmetry?¹ , ²

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ABSTRACT

We show how the usual chiral perturbation theory description of phenomenological pion physics admits an interpretation as a low-energy string-like model associated with QCD. By naive and straightforward generalization within the context of a new class of supersymmetrical models, it is shown that this string-like structure admits a 4D, $N = 1$ supersymmetrical extension. The presence of a WZNW term in the model implies modifications of certain higher order processes involving the ordinary $SU(3)$ pion octet.

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The low-energy physics of pions is often summarized by “chiral perturbation theory (CPT)” [1] in which the group manifold of $SU_L(3) \otimes SU_R(3)$ plays a critical role. We define a matrix valued field operator

$$\frac{1}{f_\pi} \Pi^i t_i \equiv \frac{1}{f_\pi} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\eta\sqrt{\frac{2}{3}} \end{pmatrix}, \quad (1)$$

where t_1, \dots, t_8 are related to Gell-Mann’s $SU(3)$ matrices and f_π is the pion decay constant. Rigid $SU_L(3) \otimes SU_R(3)$ transformations are generated on $U \equiv \exp[i\frac{1}{f_\pi}\Pi^i t_i]$ by

$$\begin{aligned} (U)' &= \exp[-i\tilde{\alpha}^i t_i] U \exp[i\alpha^i t_i] \\ &\approx U - i\tilde{\alpha}^i t_i U + iU \alpha^i t_i + \dots, \end{aligned} \quad (2)$$

where $\tilde{\alpha}^i$ and α^i are constant real parameters. The infinitesimal transformation law above implies that the infinitesimal variation $\delta\Pi^i$ takes the form

$$\delta\Pi^i = -i f_\pi \left[\tilde{\alpha}^j \left(L^{-1} \right)_j^i - \alpha^j \left(R^{-1} \right)_j^i \right] \equiv \alpha^{(A)} \xi_{(A)}^i, \quad (3)$$

where L_i^j and R_i^j are the Maurer-Cartan forms (M-C forms) defined by

$$L_i^j(\Pi) \equiv (C_2)^{-1} \text{Tr} \left[\int_0^1 dy \, t^j \left(e^{y\Delta t_i} \right) \right], \quad (4)$$

$$\Delta t_i \equiv i \frac{1}{f_\pi} [\Pi, t_i], \quad L_i^j(0) \equiv \delta_i^j, \quad R_i^j(\Pi) \equiv L_i^j(-\Pi), \quad (5)$$

where $\Pi \equiv \pi^i t_i$. As consequences of these definitions, it follows that

$$U^{-1} dU = i \frac{1}{f_\pi} d\Pi^i R_i^j t_j, \quad (dU) U^{-1} = i \frac{1}{f_\pi} d\Pi^i L_i^j t_j. \quad (6)$$

The M-C forms can also be used to directly represent the rigid $SU_L(3) \otimes SU_R(3)$ transformations as coordinate transformations of the manifold for which Π^i are considered as the coordinates. The finite version of this coordinate transformation takes the form

$$(\Pi^i)' = K^i(\Pi), \quad K^i(\Pi) \equiv \left(\exp[\alpha^{(A)} \xi_{(A)}^j \partial_j] \Pi^i \right), \quad \partial_j \equiv \frac{\partial}{\partial \Pi^j}. \quad (7)$$

These equations inform us that the coordinate transformation described by $\Pi^i \rightarrow K^i(\Pi)$ is continuously connected to the identity transformation and therefore $\xi_{(A)}^j$ correspond to a set of vectors which generate these coordinate transformations.

The leading term in the pion effective action is the well-known $SU_L(3) \otimes SU_R(3)$ invariant non-linear σ -model [1]

$$\mathcal{S}_\sigma(\Pi) = -\frac{1}{2C_2} f_\pi^2 \int d^4x \, \text{Tr}[(\partial^\mu U^{-1}) (\partial_\mu U)], \quad (8)$$

and here $\partial_{\underline{a}} \equiv \partial/\partial x^{\underline{a}}$ with $x^{\underline{a}}$ to denote the coordinates of 4D Lorentzian spacetime. The complete pion effective action $\mathcal{S}_{eff}(\Pi)$ is much more complicated than its leading term $\mathcal{S}_{\sigma}(\Pi)$. The complete form of $\mathcal{S}_{eff}(\Pi)$ may be written as a Laurent series in f_{π}^{-2}

$$\mathcal{S}_{eff}(\Pi) = \mathcal{S}_{\sigma}(\Pi) + [\mathcal{S}_{G-L}(\Pi) + \mathcal{S}_{WZNW}(\Pi)] + \dots \quad (9)$$

The first term is of order $(f_{\pi}^{-2})^{-1}$, the next term in the square brackets is of order $(f_{\pi}^{-2})^0$, etc. The second term in the series also gives the “ p^4 ” terms in S_{G-L} (that has been parameterized in a very convenient way by Gasser and Leutwyler [2] for our purposes) and S_{WZNW} (the Wess-Zumino-Novikov-Witten [3] term).

We may quickly review S_{WZNW} by writing

$$\mathcal{S}_{WZNW} = -iN_C [2 \cdot 5!]^{-1} \int d^4x \int_0^1 dy \text{Tr} \left[(\hat{U}^{-1} \partial_y \hat{U}) \widehat{\mathcal{W}}_4 \right] \quad , \quad (10)$$

$$\widehat{\mathcal{W}}_4 = \epsilon^{abcd} (\partial_{\underline{a}} \hat{U}^{-1}) (\partial_{\underline{b}} \hat{U}) (\partial_{\underline{c}} \hat{U}^{-1}) (\partial_{\underline{d}} \hat{U}) \quad , \quad (11)$$

$$\hat{U}(y) \equiv \exp[i \frac{1}{f_{\pi}} y \Pi^i t_i] \quad , \quad \hat{U}(y=1) = U(\Pi) \quad , \quad \hat{U}(y=0) = \text{I} \quad . \quad (12)$$

Upon this we may now use the chain rule to re-write this action and observe

$$\partial_{\underline{a}} \hat{U} = \left(\frac{\partial \hat{U}}{\partial \Pi^i} \right) (\partial_{\underline{a}} \Pi^i) \equiv (\partial_i \hat{U}) (\partial_{\underline{a}} \Pi^i) \quad , \quad (13)$$

$$\rightarrow (\partial_{\underline{a}} \hat{U}^{-1}) (\partial_{\underline{b}} \hat{U}) = (\partial_i \hat{U}^{-1}) (\partial_j \hat{U}) (\partial_{\underline{a}} \Pi^i) (\partial_{\underline{b}} \Pi^j) \quad , \quad (14)$$

$$\rightarrow (\partial_{\underline{a}} \hat{U}^{-1}) (\partial_{\underline{b}} \hat{U}) \equiv \hat{\mathcal{Z}}_{ij} (\partial_{\underline{a}} \Pi^i) (\partial_{\underline{b}} \Pi^j) \quad . \quad (15)$$

Using these in S_{WZNW} leads to

$$\mathcal{S}_{WZNW} = -iN_C [2 \cdot 5!]^{-1} \int d^4x \epsilon^{abcd} \beta_{i_1 i_2 i_3 i_4}(\Pi) \left(\prod_{\ell=1}^4 \partial_{\underline{a}_{\ell}} \Pi^{i_{\ell}} \right) \quad , \quad (16)$$

$$\beta_{i_1 i_2 i_3 i_4}(\Pi) \equiv \int_0^1 dy \text{Tr} \left[(\hat{U}^{-1} \partial_y \hat{U}) \hat{\mathcal{Z}}_{i_1 i_2} \hat{\mathcal{Z}}_{i_3 i_4} \right] \quad . \quad (17)$$

Some readers may find this final form of the WZNW term surprising. It is apparently often not recognized that the use of the Vainberg construction [4] as advocated by Witten allows for the particular class of extensions of \hat{U} described above. The point that is special about this class is the fact that the 4D pion fields $\Pi^i(x)$ need not depend on the “extra coordinate” y as is often assumed. As a consequence of this choice for \hat{U} , the “pullback factor” $\left(\prod_{\ell=1}^4 \partial_{\underline{a}_{\ell}} \Pi^{i_{\ell}} \right)$ is y -independent.

From our viewpoint, this class of extensions is the most natural for 4D, $N = 1$ theories. It is an entire class because we are permitted to re-define $y \rightarrow f(y)$ for

an arbitrary but analytic function $f(y)$ and we still obtain a representation of the WZNW term (after properly re-adjusting its normalization).

We will not consider the full action of Gasser and Leutwyler [2]. For simplicity we set all masses to zero and restrict ourselves purely to the pion sector. In this limit we find

$$\mathcal{S}_{G-L} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 \quad , \quad (18)$$

$$\mathcal{S}_1 = L_1 \int d^4x \operatorname{Tr}[(\partial_{\underline{a}}U^{-1})(\partial_{\underline{b}}U)] \operatorname{Tr}[(\partial^{\underline{a}}U^{-1})(\partial^{\underline{b}}U)] \quad , \quad (19)$$

$$\mathcal{S}_2 = L_2 \int d^4x \operatorname{Tr}[(\partial_{\underline{a}}U^{-1})(\partial_{\underline{b}}U)(\partial^{\underline{a}}U^{-1})(\partial^{\underline{b}}U)] \quad , \quad (20)$$

$$\mathcal{S}_3 = L_3 \int d^4x \left(\operatorname{Tr}[(\partial^{\underline{a}}U^{-1})(\partial_{\underline{a}}U)] \right)^2 \quad , \quad (21)$$

where L_1, L_2, L_3 are dimensionless numbers. We may re-write $\mathcal{S}_{(1)}$ as

$$\mathcal{S}_{(1)} = L_1 \int d^4x \operatorname{Tr}[(\partial_{\underline{a}}U^{-1})(\partial_{\underline{b}}U)] \eta^{\underline{a}\underline{c}} \eta^{\underline{b}\underline{d}} \operatorname{Tr}[(\partial_{\underline{c}}U^{-1})(\partial_{\underline{d}}U)] \quad . \quad (22)$$

Now let us also introduce irreducible projection operators for the Lorentz indices according to

$$\begin{aligned} P^{(0)} \underline{a}\underline{b}\underline{c}\underline{d} &\equiv \frac{1}{4} \eta^{\underline{a}\underline{b}} \eta^{\underline{c}\underline{d}} \quad , \quad P^{(1)} \underline{a}\underline{b}\underline{c}\underline{d} \equiv \frac{1}{2} \left[\eta^{\underline{a}\underline{c}} \eta^{\underline{d}\underline{b}} - \eta^{\underline{a}\underline{d}} \eta^{\underline{c}\underline{b}} \right] \quad , \\ P^{(2)} \underline{a}\underline{b}\underline{c}\underline{d} &\equiv \frac{1}{2} \left[\eta^{\underline{a}\underline{c}} \eta^{\underline{d}\underline{b}} + \eta^{\underline{a}\underline{d}} \eta^{\underline{c}\underline{b}} - \frac{1}{2} \eta^{\underline{a}\underline{b}} \eta^{\underline{c}\underline{d}} \right] \quad . \end{aligned} \quad (23)$$

It is a simple exercise to show that

$$P^{(I)} \underline{a}\underline{b}\underline{c}\underline{d} P^{(J)} \underline{c}\underline{d}\underline{k}\underline{l} = \delta^{IJ} P^{(J)} \underline{a}\underline{b}\underline{k}\underline{l} \quad , \quad \eta^{\underline{a}\underline{c}} \eta^{\underline{b}\underline{d}} = \sum_{I=0}^2 P^{(I)} \underline{a}\underline{b}\underline{c}\underline{d} \quad . \quad (24)$$

This last result may be substituted back into equation (22). We can carry out a similar procedure for each of the actions $\mathcal{S}_{(2)}$ and $\mathcal{S}_{(3)}$. Thus our final answer is of the form,

$$\mathcal{S}_{G-L} = \sum_{k=1}^2 \sum_{I=0}^2 \mathcal{S}_k^{(I)} \quad , \quad (25)$$

where

$$\mathcal{S}_k^{(I)} = L_k^{4(I)} \int d^4x P^{(I)} \underline{a}_1 \dots \underline{a}_4 \mathcal{J}_{i_1 i_2 i_3 i_4}^k(\mathcal{Z}) \left(\prod_{\ell=1}^4 \partial_{\underline{a}_\ell} \Pi^{i_\ell} \right) \quad , \quad (26)$$

$$\mathcal{J}_{i_1 i_2 i_3 i_4}^1 \equiv \operatorname{Tr}[\mathcal{Z}_{i_1 i_2}] \operatorname{Tr}[\mathcal{Z}_{i_3 i_4}] \quad , \quad \mathcal{J}_{i_1 i_2 i_3 i_4}^2 \equiv \operatorname{Tr}[\mathcal{Z}_{i_1 i_2} \mathcal{Z}_{i_3 i_4}] \quad , \quad (27)$$

and $\mathcal{Z}_{ij} \equiv \widehat{\mathcal{Z}}_{ij}(y=1)$. This way of writing \mathcal{S}_{G-L} will be the same as in (18) if we define $L_1^{4(0)} \equiv L_1 + L_3$, $L_1^{4(1)} \equiv L_1$, $L_1^{4(2)} \equiv L_1$, $L_2^{4(I)} \equiv L_2$. Clearly the

WZNW and G-L actions are in the same class since both contain the same pullback factor.

Notice that

$$\left[\mathcal{J}_{i_1 i_2 i_3 i_4}^k \right] \left(\prod_{\ell=1}^4 \partial_{\underline{a}_\ell} \Pi^{i_\ell} \right) = \left[\mathcal{J}_{i_1 i_2 i_3 i_4}^k \right] \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} \delta_{j_4}^{i_4} \left(\prod_{\ell=1}^4 \partial_{\underline{a}_\ell} \Pi^{j_\ell} \right) , \quad (28)$$

and the Kroneker delta factors allows us to introduce

$$\delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} \delta_{j_4}^{i_4} \equiv \sum_{(B)} \mathcal{P}_{j_1 \dots j_4}^{(B) i_1 \dots i_4} , \quad (29)$$

where $\mathcal{P}_{j_1 \dots j_4}^{(B) i_1 \dots i_4}$ are the irreducible projection operators for 4-th rank SU(3) adjoint representation tensors. Thus the most complete generalization of \mathcal{S}_{G-L} is

$$\mathcal{S}^4 = \sum_{A,B,k} \int d^4x \mathcal{L}_k^{4(A,B)} , \quad (30)$$

$$\mathcal{L}_k^{4(A,B)} = L_k^{4(A,B)} P^{(A) \underline{a}_1 \dots \underline{a}_4} \mathcal{P}_{j_1 \dots j_4}^{(B) i_1 \dots i_4} \mathcal{J}_{i_1 i_2 i_3 i_4}^{(B) k}(\mathcal{Z}) \left(\prod_{\ell=1}^4 \partial_{\underline{a}_\ell} \Pi^{j_\ell} \right) . \quad (31)$$

If $L_k^{4(A,B)} = L_k^{4(A)}$ for all values of (B) , then $\mathcal{S}^4 = \mathcal{S}_{G-L}$.

We can now comment on the entire pion effective action to all orders in f_π^{-2} . But first let's look at \mathcal{S}_σ one more time. In terms of the Z_{ij} variable the σ -model term take the form,

$$\mathcal{S}_\sigma = -\frac{1}{2} (C_2)^{-1} f_\pi^2 \int g_{ij}(\Pi) (\partial^{\underline{a}} \Pi^i) (\partial_{\underline{a}} \Pi^j) , \quad g_{ij}(\Pi) \equiv \text{Tr}[\mathcal{Z}_{ij}] . \quad (32)$$

The vector fields $\xi_{(A)}^i$ in (7) are actually Killing vectors for this metric. To all orders in f_π^{-2} we must have

$$\mathcal{S}_{eff}(\Pi) = \mathcal{S}_\sigma(\Pi) + \int d^4x \sum_{r=2}^{\infty} f_\pi^{4-2r} \mathcal{L}_r(\Pi) + \mathcal{S}_{WZNW}(\Pi) , \quad (33)$$

$$\mathcal{L}_r(\Pi) = \sum_{A,B,k} L_k^{2r(A,B)} P^{(A) \underline{a}_1 \dots \underline{a}_{2r}} \mathcal{P}_{j_1 \dots j_{2r}}^{(B) i_1 \dots i_{2r}} \left[\mathcal{J}_{i_1 \dots i_{2r}}^{(B) k} \right] \left(\prod_{\ell=1}^{2r} \partial_{\underline{a}_\ell} \Pi^{j_\ell} \right) . \quad (34)$$

Having completed all of this we see that the pion effective action is, indeed, a Laurent series in f_π^{-2} . Along this line of thought, this expansion is one in terms of the velocity “ $\partial_{\underline{a}} \Pi^i$ ” but not accelerations, etc. The quantity $g_{ij}(\Pi)$ may be regarded as a metric in the space where the Π 's are the coordinates. Similarly, $\beta_{ijkl}(\Pi)$ and $\mathcal{J}_{i_1 \dots i_{2r}}^{(B) k}(\Pi)$ are tensorial fields in this same space.

Written in this way, the pion effective action reveals itself to be the infinite dimensional inner product between the pullback factors and a set of objects that we denote by $\mathcal{M}(\Pi)$

$$\mathcal{M}(\Pi) \equiv \left\{ g_{ij}(\Pi), \beta_{ijkl}(\Pi), \mathcal{J}_{i_1 \dots i_{2r}}^{(B)k}(\Pi) \right\} . \quad (35)$$

In this present era of theoretical particle physics, a collection of this type is quite familiar. It is called a string field theory. The quantity $\mathcal{M}(\Pi)$ is the simplest representation of the QCD meson string. This has been known to specialists for a long time. It is interesting to note that chiral perturbation theory, from this view point, emerges as a manifestation of the underlying QCD meson string field.

Thus, by simply borrowing the language of string field theory, we call the leading fields of \mathcal{M} by the names in the following table

Lowest Order Fields of $\mathcal{M}(\Pi)$

“0 – mode”	$g_{ij}(\Pi)$	“QCD graviton”
“level 1 – mode”	$\beta_{ijkl}(\Pi)$	“QCD axion”
“level 1 – modes”	$\mathcal{J}_{i_1 \dots i_4}^{(B)k}(\Pi)$	“QCD NS – NS & R – R tensors”

Table I

A fundamental problem of strongly coupled low-energy QCD theory is to understand the complete spectrum of \mathcal{M} to all order and then predict the dimensionless constants $L_k^{2r(A,B)}$. At present these constants can only be measured at low orders by experiments. We find this an exceedingly beautiful geometrical structure and would like to show that it can also occur in a 4D, $N = 1$ supersymmetric theory. However, we wish to impose an analyticity condition (also called ‘holomorphy’) on *all* higher modes in our proposal of the 4D, $N = 1$ supersymmetric QCD meson string.

In a 4D, $N = 1$ supersymmetrical theory, it is natural to expect that Π^i will occur as a part of a chiral scalar supermultiplet that we denote by $\Phi^I(\theta, \bar{\theta}, x)$. Since

$$\bar{D}_{\dot{\alpha}} \Phi^I = 0 \quad , \quad (36)$$

we think it is natural to impose some type of holomorphy on the supersymmetric analog of $\mathcal{M}(\Pi)$. It turns out that it is impossible to do this on the supersymmetric analog of $g_{ij}(\Pi)$.

In 1984, [5] we began to wonder if it might be possible to impose holomorphy on the higher modes. In 1995, we returned our attention to this problem and found a remarkable solution [6]. In order to show the existence of this solution we had to

construct a new type of 4D, $N = 1$ supersymmetric non-linear σ -model. This explicit solution makes use of a little known representation of 4D, $N = 1$ supersymmetry called “the non-minimal multiplet” (which first appeared in a 1981 paper [7]) or “complex linear” multiplet. Additionally the model also uses chiral multiplets. Accordingly, we call this class of models “CNM - models” (for chiral-nonminimal models).

The non-minimal multiplet, like the chiral multiplet, only describes physical helicities $(0^+, 0^-, 1/2)$ on-shell and is defined by the equation

$$\overline{D}^2 \Sigma^I = 0 \quad . \quad (37)$$

Since we are now moving on to 4D, $N = 1$ supersymmetry, in the next part of this presentation, we will review some basic facts about these multiplets.

The chiral multiplet was first discovered by Gol’fand and Likhtman [8] and then by Wess and Zumino [9]. Its simplest action is

$$\begin{aligned} \mathcal{S}_C &= \int d^4x d^2\theta d^2\bar{\theta} \overline{\Phi} \Phi \\ &\equiv \int d^4x \left[-\frac{1}{2}(\partial^{\underline{a}} \overline{A})(\partial_{\underline{a}} A) - i \overline{\psi}^{\dot{\alpha}} \partial_{\underline{a}} \psi^{\alpha} + \overline{F} F \right] \quad . \end{aligned} \quad (38)$$

At this point we will say something about notation. It has long been the convention of ‘*Superspace*’ to define

$$x^{\underline{a}} \equiv \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad . \quad (39)$$

We have always chosen to regard the coordinate of space-time as being parametrized by a hermitian two-by-two matrix⁴ $x^{\underline{a}}$. This convention implies that $\partial_{\underline{a}}$ is also a matrix and thus the form of the component level action above. On-shell we find $\overline{D}^2 \overline{\Phi} = 0$ as the superfield equation of motion, so that

$$\partial^{\underline{a}} \partial_{\underline{a}} A = 0 \quad , \quad -i \partial_{\underline{a}} \psi^{\alpha} = 0 \quad , \quad F = 0 \quad , \quad (40)$$

emerge as the dynamical equations of motion. Clearly the only physical degrees of freedom are those of A and ψ^{α} . The field F has an algebraic equation of motion and is therefore called an “auxiliary field.”

⁴This is highly amusing in light of the recent suggestion of M-theory as M(atrix) theory where spacetime there also is described by matrices.

At this point, we may take the opportunity to describe an aspect of the theory that will be relevant later. We may modify the action in (38) to the form

$$\begin{aligned}\mathcal{S}_{C(n)} &= \frac{1}{\Lambda^n} \int d^4x d^2\theta d^2\bar{\theta} \bar{\Phi} \left(\partial^{\underline{d}} \partial_{\underline{d}} \right)^n \Phi \\ &\equiv \frac{1}{\Lambda^n} \int d^4x \left[\frac{1}{2} (\bar{A} (\partial^{\underline{d}} \partial_{\underline{d}})^{n+1} A) - i \bar{\psi}^{\dot{\alpha}} (\partial^{\underline{d}} \partial_{\underline{d}})^n \partial_{\underline{a}} \psi^{\alpha} + \bar{F} (\partial^{\underline{d}} \partial_{\underline{d}})^n F \right] ,\end{aligned}\quad (41)$$

where n is any positive integer and Λ has the units of (mass)². It is clear that the concept of F being an ‘‘auxiliary field’’ no longer applies to this action since F now possesses a dynamical equation of motion. In general, if one considers an action of the form

$$\mathcal{S}_{C-H.D.} = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{F} \left(\Phi, \bar{\Phi}, D_{\underline{A}} \Phi, D_{\underline{A}} \bar{\Phi}, \dots \right) , \quad (42)$$

it is usually the case that such an expression has the consequence

$$\frac{\delta \mathcal{S}_{C-H.D.}}{\delta F} = 0 \quad , \quad \rightarrow \text{dynamical equation for } F \quad . \quad (43)$$

On the other hand, the simplest action for the non-minimal multiplet is

$$\begin{aligned}\mathcal{S}_{NM} &= - \int d^4x d^2\theta d^2\bar{\theta} \bar{\Sigma} \Sigma \\ &= \int d^4x \left[-\frac{1}{2} (\partial^{\underline{a}} \bar{B}) (\partial_{\underline{a}} B) - i \bar{\zeta}^{\alpha} \partial_{\underline{a}} \zeta^{\dot{\alpha}} - \bar{H} H + 2 \bar{p}^{\underline{a}} p_{\underline{a}} \right. \\ &\quad \left. + \beta^{\alpha} \rho_{\alpha} + \bar{\beta}^{\dot{\alpha}} \bar{\rho}_{\dot{\alpha}} \right] .\end{aligned}\quad (44)$$

On-shell we find the superfield equation of motion $\bar{D}_{\dot{\alpha}} \bar{\Sigma} = 0$ or at the component level

$$\partial^{\underline{d}} \partial_{\underline{d}} B = 0 \quad , \quad -i \partial_{\underline{a}} \bar{\zeta}^{\alpha} = 0 \quad , \quad H = 0 \quad , \quad p_{\underline{a}} = 0 \quad , \quad \bar{\rho}_{\dot{\alpha}} = 0 \quad , \quad \beta_{\alpha} = 0 \quad , \quad (45)$$

and since $p_{\underline{a}} \neq \bar{p}_{\underline{a}}$, the action contains 12 bosons and 12 fermions.

The two action S_C and S_{NM} provide an example of two theories that are related to each other by ‘Poincaré duality.’ This is most easily seen by comparing the constraints and equations with those of electromagnetism

	Constraint	Equation of Motion
E. & M.	$d F = 0$	$d^* F = 0$
Chiral SF	$\bar{D}_{\dot{\alpha}} \Phi = 0$	$D^2 \Phi = 0$
Nonminimal SF	$D^2 \bar{\Sigma} = 0$	$\bar{D}_{\dot{\alpha}} \bar{\Sigma} = 0$

Table II

Another example of Poincaré duality can be seen by considering the theory of a massless scalar versus that of the “notoph.” The action for an ordinary massless scalar is provided by

$$\mathcal{S}_S = - \int d^4x \left[\frac{1}{2} F_{\underline{a}} F_{\underline{a}} \right] \quad , \quad F_{\underline{a}} \equiv \partial_{\underline{a}} \varphi \quad . \quad (46)$$

It is seen that $F_{\underline{a}}$ satisfies a constraint and has an equation of motion that are given in the following table

Bianchi identity	Equation of motion
$\partial^{\underline{a}} F_{\underline{a}} = 0$	$\partial_{\underline{a}} F_{\underline{b}} - \partial_{\underline{b}} F_{\underline{a}} = 0$

Table III

In comparison, for the “notoph” [11] we also see the same phenomenon

$$\mathcal{S}_N = \int d^4x \left[\frac{1}{2} H_{\underline{a}} H_{\underline{a}} \right] \quad , \quad H_{\underline{a}} \equiv \frac{1}{2} \epsilon_{\underline{a}\underline{b}\underline{c}\underline{d}} \partial^{\underline{b}} \partial^{\underline{c}} \varphi^{\underline{d}} = \frac{1}{3!} \epsilon_{\underline{a}\underline{b}\underline{c}\underline{d}} H^{\underline{bcd}} \quad , \quad (47)$$

where the constraint and equation of motion for $H_{\underline{a}}$ are found to be as in the following table

Bianchi identity	Equation of motion
$\partial_{\underline{a}} H_{\underline{b}} - \partial_{\underline{b}} H_{\underline{a}} = 0$	$\partial^{\underline{a}} H_{\underline{a}} = 0$

Table IV

We see exactly the exchange of the constraint with the equation of motion and vice-versa. As well the form of S_N is the same as that of S_S with the only difference being a sign (c.f. S_C and S_{NM}). Thus, $(\Phi^{\mathbf{I}}, \Sigma^{\mathbf{I}})$ constitute a Poincaré dual pair.

Now we must realize the symmetry $\text{SU}_L(3) \otimes \text{SU}_R(3)$ within the context of a 4D, $N = 1$ supersymmetrical theory. For this purpose, we introduce a mapping operation denoted by \mathcal{G}_S^C with the property

$$\mathcal{G}_S^C : \exp[\pm i \frac{1}{f_\pi} \Pi^i t_i] \rightarrow \exp \left[\frac{\pm \Phi^{\mathbf{I}} t_{\mathbf{I}}}{f_\pi \cos(\gamma_S)} \right] \quad . \quad (48)$$

In this expression $\Phi^{\mathbf{I}}$ are eight chiral superfields, $t_{\mathbf{I}}$ are exactly the same matrices which appeared in the non-supersymmetrical theory and γ_S is a mixing angle about which we shall say more later. Note that since $\Phi^{\mathbf{I}} \neq \overline{\Phi}^{\mathbf{I}}$ it follows that the superfield U obeys

$$U^\dagger \neq U^{-1} \quad . \quad (49)$$

Now the realization of the $SU_L(3) \otimes SU_R(3)$ symmetry proceeds exactly as before using equation (2). As before, $\tilde{\alpha}^I$ and α^I are still real constants. Also similar to before we find

$$\delta\Phi^I = -i [f_\pi \cos(\gamma_S)] [\tilde{\alpha}^J (L^{-1})^I_J - \alpha^J (R^{-1})^I_J] \equiv \alpha^{(A)} \xi_{(A)}^I(\Phi) \quad . \quad (50)$$

So Φ^I transforms infinitesimally like a coordinate. For finite values of $\tilde{\alpha}^I$ and α^I there is induced a superfield coordinate transformation

$$(\Phi^I)' = K^I(\Phi) \quad , \quad K^I(\Phi) \equiv \left(\exp[\alpha^{(A)} \xi_{(A)}^L \partial_L] \Phi^I \right) \quad , \quad \partial_L \equiv \frac{\partial}{\partial \Phi^L} \quad . \quad (51)$$

Thus $\xi_{(A)}^I(\Phi)$ is the superfield Killing vector generator. Once more $(L)_I^J$ and $(R)_I^J$ are Maurer-Cartan forms but these are also now superfields. The only difference in their superfield definitions of the M-C forms is that we must use Δ defined by

$$\Delta t_I \equiv [f_\pi \cos(\gamma_S)]^{-1} [\Phi, t_I] \quad , \quad \Phi \equiv \Phi^L t_L \quad . \quad (52)$$

In 1984 [12] we wrote the following 4D, $N = 1$ nonlinear σ -model superfield action

$$\mathcal{S}_{\text{KVM}} = \left\{ \int d^4x \left[\int d^2\theta d^2\bar{\theta} \bar{\Phi}^I \partial_I + \frac{1}{4} \int d^2\theta W^{\alpha I} \partial_I W_\alpha^K \partial_K \right] H(\Phi) + \text{h.c.} \right\} \quad , \quad (53)$$

to describe ‘‘Kahlerian Vector Multiplet’’ models. Today this is widely referred to as the ‘Seiberg-Witten effective action’ after their discovery [13] that for a special choice of the function $H(\Phi)$ (related to elliptical curves), this action describes the leading term of the effective action of the $N = 2$ vector multiplet. This action actually does possess a 4D, $N = 2$ supersymmetry invariance. However, if we regard Φ^I here as a coordinate, then clearly W_α^I , the vector multiplet field strength superfield, must transform as a 1-form in this space in order for this action to be invariant,

$$(\Phi^I)' = K^I(\Phi) \rightarrow (W_\alpha^I)' = W_\alpha^J (\partial_J K^I) \quad . \quad (54)$$

This strongly suggests that in the present context Σ^I should transform in the same manner as W_α^I in the 4D, $N = 2$ supersymmetric Yang-Mills theory,

$$(\Sigma^I)' = \Sigma^J (\partial_J K^I) \quad . \quad (55)$$

Thus, motivated by our experience with the KVM action, we have proposed the ‘‘CNM nonlinear σ -model’’ action

$$\mathcal{S}_\sigma^{CNM}(\Phi, \Sigma) = \left(\frac{1}{c_2} \right) f_\pi^2 \cos^2(\gamma_S) \int d^4x d^2\theta d^2\bar{\theta} \left[1 - \Sigma^I \frac{\partial}{\partial \Phi^I} \bar{\Sigma}^K \frac{\partial}{\partial \bar{\Phi}^K} \right] K(\Phi, \bar{\Phi}) \quad , \quad (56)$$

and if $K(\Phi, \bar{\Phi})$ (the Kähler potential) is invariant under the transformation generated by the superfield Killing vector $\xi_{(A)}^I(\Phi)$, then $SU_L(3) \otimes SU_R(3)$ invariance of the action follows as a consequence. There are many such functions [14], with one being suggested by Pernici and Riva (1986) [15] as

$$\begin{aligned} K(\Phi, \bar{\Phi}) &\equiv Tr \left[U(\Phi) U^\dagger(\Phi) \right] \quad , \\ \rightarrow g_{I\bar{J}}(\Phi, \bar{\Phi}) &= \partial_I \bar{\partial}_{\bar{J}} Tr \left[U(\Phi) U^\dagger(\Phi) \right] \quad . \end{aligned} \quad (57)$$

Notice that this choice of Kahler potential is invariant only under $SU_L(3) \otimes SU_R(3)$ not under $[SU_L(3) \otimes SU_R(3)]^c$, the complexification.

The quantity $g_{I\bar{J}}(\Phi, \bar{\Phi})$ is clearly not a holomorphic function and generalizes $g_{ij}(\Pi)$, the ‘QCD meson-string zero mode.’ We believe that this must be true in all 4D, $N = 1$ supersymmetrical QCD meson-string models otherwise the complexified group $[SU_L(3) \otimes SU_R(3)]^c$ would appear as a symmetry at lowest order in f_π . Henceforth, we shall refer to $g_{I\bar{J}}$ as “the 4D, $N = 1$ supersymmetric QCD meson-string zero mode.” From its relation to $K(\Phi, \bar{\Phi})$, we see that it describes a Kähler geometry as it should.

There remains the task of constructing all analogs of the higher order ‘fields’ $\beta_{i j k l}$ and $\mathcal{J}_{i_1, \dots, i_{2r}}^{(B)k}$. Let us denote their superfield extensions by

$$\beta_{I J K L} \quad , \quad \mathcal{J}_{I_1 \dots I_{2r}}^{(B)k} \quad , \quad (58)$$

and on these we wish to impose the chirality (holomorphy) conditions

$$\begin{aligned} \bar{D}_\alpha \beta_{I J K L} &= \bar{D}_\alpha \mathcal{J}_{I_1 \dots I_{2r}}^{(B)k} = 0 \quad , \\ \rightarrow \bar{\partial}_M \beta_{I J K L} &= \bar{\partial}_M \mathcal{J}_{I_1 \dots I_{2r}}^{(B)k} = 0 \quad . \end{aligned} \quad (59)$$

Our motivations for doing this are two-fold;

- (a.) The higher derivative terms are to be regarded as interactions for the physical states. For non-derivative interactions, a holomorphic superpotential $W(\Phi)$ is usually introduced. We want the derivative terms also determined by holomorphic tensors .
- (b.) In addition to the physical fields $\{\mathcal{X}\} \equiv (A, B, \psi_\alpha, \zeta_\alpha)$ there are lots of auxiliary fields $\{\mathcal{Y}\} \equiv (F, H, p_{\underline{a}}, \beta_\alpha, \rho_\alpha)$. The conditions that

$$\frac{\delta \mathcal{S}_{C-H.D.}}{\delta \mathcal{Y}} = 0 \quad , \quad \rightarrow \text{algebraic equations for } \mathcal{Y} \quad , \quad (60)$$

are satisfied as a consequence of the holomorphy conditions!

This is true even though \mathcal{S}_{eff}^{CNM} contains derivatives to all powers.

This last property is so striking that we have named it “auxiliary freedom.” In fact, it was auxiliary freedom of higher derivative supersymmetric actions about which we began to wonder in 1984. This is in stark contrast to the result in equation (43).

We must still face the task of constructing the actions containing β_{IJKL} and $\mathcal{J}_{I_1 \dots I_{2r}}^{(B)k}$. Fortunately, this can be described in a single step! The trick is to extend the definition of the map \mathcal{G}_S^C . We do this according to the following rules;

$$(b.) \quad \mathcal{G}_S^C : \left(\prod_{\ell=1}^{2r} \partial_{\underline{a}_\ell} \Pi^{j_\ell} \right) \rightarrow C_{\alpha_1 \alpha_{2r}} \left(\overline{D}_{\dot{\alpha}_1} \Sigma^{I_1} \right) \left(\overline{D}_{\dot{\alpha}_{2r}} \Sigma^{I_{2r}} \right) \left(\prod_{\ell=2}^{2r-1} \partial_{\underline{a}_\ell} \Phi^{j_\ell} \right) \quad , \quad (61)$$

$$(c.) \quad \mathcal{G}_S^C : \int d^4x \rightarrow \int d^4x d^2\theta \quad , \quad (62)$$

$$(d.) \quad \mathcal{G}_S^C : L_k^{2r(A,B)} \rightarrow l_k^{2r(A,B)} + im_k^{2r(A,B)} \quad . \quad (63)$$

In this last rule, the dimensionless parameters $l_k^{2r(A,B)}$ and $m_k^{2r(A,B)}$ are real. Rule (a.) was given when we first discussed how to generalize the group elements to superfields in equation (48). Let us comment on each of these in turn.

Rule (b.) essential follows from dimensional analysis, Lorentz covariance and the fact that Φ^I transforms like a coordinate and Σ^I transforms like a 1-form.

Rule (c.) follows upon observing that

$$\overline{D}_{\dot{\alpha}} \mathcal{G}_S^C : \left(\prod_{\ell=1}^{2r} \partial_{\underline{a}_\ell} \Pi^{j_\ell} \right) = 0 \quad , \quad (64)$$

and using the holomorphy of β_{IJKL} and $\mathcal{J}_{I_1 \dots I_{2r}}^{(B)k}$.

Rule (d.) recalls an analogy to supersymmetric Yang-Mills theory where

$$\mathcal{S}_{YM} = Tr \left[\frac{1}{8g^2} \int d^4x d^2\theta W^\alpha W_\alpha + \text{h.c.} \right] \quad . \quad (65)$$

The constant g^2 may be considered as being complex

$$\frac{1}{g^2} = \frac{1}{e^2} + i \frac{\theta}{4\pi} \quad . \quad (66)$$

So we propose the constants in the pion effective action can also acquire imaginary parts under \mathcal{G}_S^C . Like their analogs in supersymmetric Yang-Mills theory, however, such terms are odd under parity transformations.

We are now able to write the entirety of a 4D, $N = 1$ supersymmetric extension in a single step as

$$\begin{aligned} \mathcal{S}_{eff}^{CNM}(\Phi, \Sigma) \equiv & \mathcal{S}_\sigma^{CNM} + \left\{ \mathcal{G}_S^C : \left[\int d^4x \sum_{r=2}^{\infty} f_\pi^{4-2r} \mathcal{L}_r(\Pi) \right] + \text{h. c.} \right\} \\ & + \left\{ \mathcal{G}_S^C : \left[\mathcal{S}_{WZNW}(\Pi) \right] + \text{h. c.} \right\} , \end{aligned} \quad (67)$$

and every term in the non-supersymmetric action goes directly over to superspace! Like its non-supersymmetric analog, this action is of the form of an infinite inner product of superspace pullbacks with the collection of objects $\mathcal{M}_s(\Phi)$

$$\mathcal{M}_s(\Phi) \equiv \left\{ K(\Phi, \overline{\Phi}), \beta_{IJKL}(\Phi), \mathcal{J}_{I_1 \dots I_{2r}}^{(B)k}(\Phi) \right\} , \quad (68)$$

which we obviously propose as the 4D, $N = 1$ supersymmetric QCD meson string. In the CNM approach, the elements of \mathcal{M}_s are

$$K(\Phi, \overline{\Phi}) \equiv \text{Tr} \left[U U^\dagger \right] , \quad (69)$$

$$\beta_{IJKL}(\Phi) \equiv \frac{1}{4!} \int_0^1 dy \text{Tr} \left[(\hat{U}^{-1} \partial_y \hat{U}) \hat{\mathcal{Z}}_{[IJ]} \hat{\mathcal{Z}}_{|KL|} \right] , \quad (70)$$

$$\mathcal{J}_{I_1 \dots I_{2r}}^{(B)k}(\Phi) \equiv \mathcal{P}_{L_1 \dots L_{2r}}^{(B) K_1 \dots K_{2r}} \text{Tr}^k \left[\mathcal{Z}_{K_1 K_2} \cdots \mathcal{Z}_{K_{2r-1} K_{2r}} \right] , \quad (71)$$

and in the final one of these results Tr^k denotes the distinct ways of taking traces over the r distinct \mathcal{Z} -factors (c.f. (27)). Note that in terms of Kähler geometry, β_{IJKL} is a $(4,0)$ tensor. Similarly $\mathcal{J}_{I_1 \dots I_{2r}}^{(B)k}$ is a $(2r,0)$ tensor.

It is apparent that the proposal implies that \mathcal{M}_s depends *solely* on the chiral superfields Φ^I and not at all on the nonminimal superfields Σ^I . The role of the nonminimal superfields is restricted to their appearing via the superspace pullback factors. Thus they provide a means of “projecting” the 4D, $N = 1$ supersymmetric QCD meson string onto 4D, $N = 1$ superspace.

This entire action possesses rigid $\text{SU}_L(3) \otimes \text{SU}_R(3)$ invariance. It is therefore of interest to obtain the superfield currents which correspond to this symmetry. One way to obtain these is to gauge this symmetry group using the standard superfield approach. This would require the introduction of $\text{SU}(3)$ matrix-valued gauge superfields $V_{(L)}$ and $V_{(R)}$ into the action \mathcal{S}_{eff}^{CNM} . However, this requires also the solution to the problem of gauging the superfield WZNW term (which we have not yet obtained).

So in order to at least obtain a preliminary view of these superfield currents, we will gauge these symmetries in \mathcal{S}_σ^{CNM} via the modification

$$K(\Phi, \bar{\Phi}) \rightarrow Tr \left[e^{V_{(L)}} U e^{-V_{(R)}} U^\dagger \right] . \quad (72)$$

We now calculate $\delta \mathcal{S}_\sigma^{CNM}(\Phi, \Sigma)/\delta V_{(L)}^I$ and $\delta \mathcal{S}_\sigma^{CNM}(\Phi, \Sigma)/\delta V_{(R)}^I$ which defines two currents (with $n_0 \equiv (C_2)^{-1} f_\pi^2 \cos^2(\gamma_S)$)

$$J_I^{(L)} \equiv n_0 \left[1 - \Sigma^K \bar{\Sigma}^L \partial_K \bar{\partial}_L \right] Tr \left[U^\dagger t_I U \right] , \quad (73)$$

$$J_I^{(R)} \equiv -n_0 \left[1 - \Sigma^K \bar{\Sigma}^L \partial_K \bar{\partial}_L \right] Tr \left[U t_I U^\dagger \right] , \quad (74)$$

that are obtained after setting the gauge superfields to zero in the variations. These left and right currents can then be used to define superfields that contain the component level vector and axial vector currents

$$J^{I(v)} \equiv \frac{1}{\sqrt{2}} [J_I^{(L)} + J_I^{(R)}] , \quad J^{I(a)} \equiv \frac{1}{\sqrt{2}} [J_I^{(L)} - J_I^{(R)}] . \quad (75)$$

These expressions open the way to study the superfield current algebra associated with the realization of the symmetry group. The ordinary component level currents associated with the symmetries are obtained uniformly by the rule $J_a \equiv ([D_a, \bar{D}_a] J)|$.

The reader will recall the presence of γ_S (the mixing angle) throughout the discussion. We can now discuss its role. We identify the physical pion SU(3) octet $\Pi^I(x)$ through the definitions

$$\Phi^I| = A^I(x) = \mathcal{A}^I(x) + i \left[\Pi^I(x) \cos(\gamma_S) + \Theta^I(x) \sin(\gamma_S) \right] , \quad (76)$$

$$\Sigma^I| = B^I(x) = \mathcal{B}^I(x) + i \left[-\Pi^I(x) \sin(\gamma_S) + \Theta^I(x) \cos(\gamma_S) \right] . \quad (77)$$

The lowest components of the superfields Φ^I and Σ^I endow the model with two scalar spin-0 degrees of freedom and two pseudoscalar degrees of freedom. In general there can occur mixing among these states. In order to allow for this (in a parity conserving manner) we introduce the fields as above. (For the sake of simplicity, we only consider mixing of the pseudoscalar states. Momentarily, it will be clear why this is done.)

In addition to the dynamical spin-0 degrees of freedom, this model also possesses an SU(3) flavor octet of Dirac spin-1/2 degrees of freedom (denoted by $\ell^I(x)$) that we call ‘‘pionini.’’ These occur in the superfields as $(\psi_\alpha^I(x), \zeta_\alpha^I(x))$

$$\ell^I(x) \equiv \begin{pmatrix} \psi_\alpha^I \\ \zeta_\alpha^I \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} , \quad \begin{aligned} \psi_\alpha^I &= \frac{1}{2}(I + \gamma^5)\ell \\ \zeta_\alpha^I &= \frac{1}{2}(I - \gamma^5)\ell \end{aligned} . \quad (78)$$

The CNM class of models is heterodexterous, i.e. all left-handed dynamical spinors reside in nonminimal multiplets and right-handed dynamical spinors reside in chiral multiplets. The remaining fields of the model are the auxiliary fields. We will *not* here give their definitions in terms of D -operators acting on superfields. Suffice it to say that the most useful definitions are slightly different from those associated with the free chiral and nonminimal multiplets.

A hallmark of the ordinary WZNW term is that it contains a five pion field operator vertex as its leading term. Here it is relevant to know if this vertex appears in the supersymmetric generalization in (67). If we demand that the supersymmetric WZNW term contains exactly this vertex, then it is necessary to impose the condition that $\sin^2(2\gamma_S) \neq 0$. In other words in order to produce the proper dynamics for the pion contained in this model, mixing is absolutely critical.

Now of course all of this is very fanciful and may be regarded as purely a mathematical exercise. But let us go the one remaining step. Having embedded the ordinary pion effective action into a 4D, $N = 1$ model, we can ask if this causes any modifications in the purely pion sector of the theory? The answer is, ‘Yes.’

If we retain only terms in $\mathcal{S}_\sigma^{CNM}(\Phi, \Sigma)$ that depend solely on the $\Pi^i(x)$ field operator, we find (by lim below, we mean keep only the SU(3) pion field dependence)

$$\tilde{\mathcal{S}}_\sigma(\Pi) \equiv \left[\lim \mathcal{S}_\sigma^{CNM}(\Phi, \Sigma) \right] = \mathcal{S}_\sigma(\Pi) + \sin^2(\gamma_S) \left[S_1(\Pi) + S_2(\Pi) \right] , \quad (79)$$

where $\mathcal{S}_\sigma(\Pi)$ is the usual pion model nonlinear σ -model of (8). However, there are the “extra” terms whose explicit forms are given by

$$S_1(\Pi) = -\left(\frac{f_\pi^2}{2C_2}\right) \int d^4x \text{Tr} \left[\left(\frac{\partial U^\dagger}{\partial \Pi^r} \right) \left(\frac{\partial^2 U}{\partial \Pi^j \partial \Pi^r} \right) + \text{h.c.} \right] \Pi^i (\partial^a \Pi^j) (\partial_a \Pi^k) , \quad (80)$$

$$S_2(\Pi) = -\left(\frac{f_\pi^2}{2C_2}\right) \int d^4x \text{Tr} \left[\left(\frac{\partial^2 U^\dagger}{\partial \Pi^r \partial \Pi^j} \right) \left(\frac{\partial^2 U}{\partial \Pi^k \partial \Pi^r} \right) \right] \Pi^i \Pi^k (\partial^a \Pi^j) (\partial_a \Pi^l) . \quad (81)$$

The lowest order effects of these “extra” terms can be seen by extracting the form of vertices that have four powers of SU(3) pion field operators and two spacetime derivatives⁵. Terms of this type arise from \mathcal{S}_σ , \mathcal{S}_1 and \mathcal{S}_2 and from no other terms in the action (67). By expanding out $\exp[i\frac{1}{f_\pi}\Pi^i t_i]$ to appropriate orders, we find the required terms take the form

$$-\left(\frac{1}{12 C_2 f_\pi^2}\right) G_{ijkl} \int d^4x \Pi^i \Pi^k (\partial^a \Pi^j) (\partial_a \Pi^l) , \quad (82)$$

⁵Contrary to appearances, \mathcal{S}_1 does not lead to a vertex containing three SU(3) pion field operators.

where the “coupling constant” has the explicit representation,

$$\begin{aligned}
G_{ijkl} = & [1 + \frac{1}{2} \sin^2(\gamma_S)] \text{Tr} \left[\{ t_i, t_j \} \{ t_k, t_l \} \right] \\
& - [1 - \sin^2(\gamma_S)] \text{Tr} \left[\{ t_i, t_k \} \{ t_j, t_l \} \right] \\
& - \frac{1}{2} [1 + 2 \sin^2(\gamma_S)] \text{Tr} \left[[t_i, t_j] [t_k, t_l] \right] .
\end{aligned} \tag{83}$$

It is a simple matter to reduce this form of the coupling constant to another form involving the f and d coefficients of $\text{SU}(3)$. This result for $\gamma_S = 0$ is what follows from (8). The two extra actions S_1 and S_2 are present because the WZNW term at higher order imposes the condition that $\sin^2(\gamma_S)$ cannot be equal to zero! These extra terms are suppressed by $\sin^2(\gamma_S)$ and if γ_S is very small then the strength of these additional pion interaction terms is very, very weak...like most people’s sense of smell.

It would be naive to claim that this really happens in nature. Life is almost never so simple. We would have to be incredibly fortunate to live in a universe with broken supersymmetry where nevertheless, higher order $\text{SU}(3)$ pion octet measurements could be made to detect the $\sin^2(\gamma_S)$ dependence of G_{ijkl} . Still it is amusing to contemplate how perilously close this supersymmetric model comes to making an actual prediction. What we have shown is that with the assumptions;

* $\text{SU}_L(3) \otimes \text{SU}_R(3)$ symmetry,

*a complete CPT model with higher derivative terms,

*4D, $N = 1$ supersymmetry,

*holomorphy of all non-zero modes of the supersymmetric QCD
meson string,

we are led to the following spectrum and dynamics,

*1 Nambu-Goldstone boson + 3 quasi-Nambu-Goldstone bosons,

*a Dirac Nambu-Goldstino,

*no propagating auxiliary fields and

*additional pion interactions that are characterized by a nonvanishing
mixing angle γ_S .

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